Overview of Complex Analysis (Gamelin)

Brett Saiki

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This is a condensed version of Theodore W. Gamelin's *Complex Analysis* containing only definitions, propositions, theorems, etc. For proofs and detailed explanations, refer to the actual text.

1 The Complex Plane and Elementary Functions

1.1 Complex Numbers

Definition. A **complex number** is an expression of the form $z = x + iy$ where x and y are real numbers. The component x the **real part** of z and y the **imaginary part** of z. We will denote these with

$$x = \operatorname{Re} z \qquad\qquad y = \operatorname{Im} z$$

The set of all complex numbers is called the **complex plane** and denote it with $\mathbb{C}$. There exists a one-to-one correspondence between the complex numbers and points in the Euclidean plane $\mathbb{R}^2$.

$$z = x + iy \longleftrightarrow (x, y)$$

The real numbers correspond to the x-axis in the Euclidean plane while the **purely imaginary numbers** correspond to the y-axis and are of the form iy. The purely imaginary numbers form the **imaginary axis** $i\mathbb{R}$.

Definition. We add complex numbers by adding their real and imaginary parts separately.

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$

Thus, $\operatorname{Re}(z+w) = \operatorname{Re} z + \operatorname{Re} w$ and $\operatorname{Im}(z+w) = \operatorname{Im} z + \operatorname{Im} w$. The addition of complex numbers corresponds to the addition of vectors in the Euclidean plane.

Definition. The **modulus** of a complex number $z = x + iy$ is the length $\sqrt{x^2 + y^2}$ of the corresponding vector in the Euclidean plane. The modulus is also called the **absolute value** of z.

Definition. The triangle inequality also applies to complex numbers:

$$|z + w| \leq |z| + |w| \qquad z, w \in \mathbb{C}$$

And so is the inequality involving subtraction:

$$|z - w| \geq |z| - |w| \qquad z, w \in \mathbb{C}$$

Definition. Unlike for vectors in $\mathbb{R}^2$, multiplication is well-defined for complex numbers:

$$(x + iy)(u + iv) = xu - yv + i(xv + yu)$$

The usual laws of multiplication hold true:

$$(z_1 z_2) z_3 = z_1 (z_2 z_3) \qquad \text{(associative law)}$$
$$z_1 z_2 = z_2 z_1 \qquad \text{(commutative law)}$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \text{(distributive law)}$$

Definition. Every complex number $z \neq 0$ has a multiplicative inverse $1/z$ which is given explicity by

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}, \qquad z = x + iy \in \mathbb{C}, z \neq 0$$

Definition. The **complex conjugate** of a complex number $z = x + iy$ is defined to be $z = x - iy$. Geometrically speaking, it is the reflection of z across the x-axis in the Euclidean plane.

The following are useful identities involving conjugates:

$$\begin{aligned}
\overline{\overline{z}} &= z, && z \in \mathbb{C} \\
\overline{z+w} &= \overline{z} + \overline{w}, && z, w \in \mathbb{C} \\
\overline{zw} &= \overline{z}\overline{w}, && z, w \in \mathbb{C} \\
|z| &= |\overline{z}|, && z \in \mathbb{C} \\
|z|^2 &= z\overline{z}, && z \in \mathbb{C}
\end{aligned}$$

We can rewrite the $1/z$ in terms of the complex conjugate of z:

$$1/z = \overline{z}/|z|^2, \qquad z \in \mathbb{C}, z \neq 0$$

The real and imaginary parts of z can be recovered using complex conjugates:

$$\operatorname{Re} z = (z + \overline{z})/2, \qquad \operatorname{Im} z = (z - \overline{z})/2i, \qquad z \in \mathbb{C}$$

From $|zw|^2 = (zw)(\overline{zw}) = (z\overline{z})(w\overline{w}) = |z|^2|w|^2$, we obtain:

$$|zw| = |z||w|$$

Definition. A **complex polynomial of degree** $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \qquad z \in \mathbb{C}$$

where $a_0, \ldots, a_n$ are complex numbers and $a_n \neq 0$.

Fundamental Theorem of Algebra. Every polynomial $p(z)$ of degree $n \geq 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$$

where the z_j's are unique and $m_j \geq 1$. This factorization is unique, up to a permutation of the factors.

1.2 Polar Representation

Definition. Since any point (x, y) in the plane can be represented by polar coordinates r and θ where $r = \sqrt{x^2 + y^2}$ and θ is the angle subtended by (x, y) and the x-axis, we can also express complex numbers using polar coordinates.

$$z = x + iy = r(\cos\theta + \sin\theta)$$

Here $r = |z|$ is the modulus of z. We define the **argument** of z to be the angle θ and we write

$$\theta = \arg z$$

Thus $\arg z$ is a mutli-valued function, defined for $z \neq 0$. The **principal value of** $\arg z$, denoted $\operatorname{Arg} z$, is the value of θ within $-\pi < \theta \leq \pi$. The values of $\arg z$ are obtained by adding integer multiples of 2π to $\operatorname{Arg} z$:

$$\arg z = \{\operatorname{Arg} z + 2\pi k : k = 0, \pm 1, \pm 2, \ldots\}, \qquad z \neq 0$$

Definition. It is often more convenient to write

$$z = re^{i\theta} \qquad r = |z|,\ \theta = \arg z$$

This representation is the **polar form** of z. Since $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{i\theta}$ is also 2π-periodic, so

$$e^{i(\theta+2\pi m)} = e^{i\theta}, \qquad m = 0, \pm1, \pm2, \dots$$

and

$$e^{2\pi mi} = 1, \qquad m = 0, \pm1, \pm2, \dots$$

Some useful identites involving polar form:

$$\begin{aligned} |e^{i\theta}| &= 1 \\ \overline{e^{i\theta}} &= e^{-i\theta} \\ 1/e^{i\theta} &= e^{-i\theta} \end{aligned}$$

Definition. An important property of the exponential function is the **addition formula**:

$$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}, \qquad -\infty < \theta, \varphi < \infty$$

We can rewrite the previous equations in terms of the argument function

$$\begin{aligned} \arg \overline{z} &= -\arg z \\ \arg(1/z) &= -\arg z \\ \arg(z_1 z_2) &= \arg z_1 + \arg z_2 \end{aligned}$$

Definition. If we let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then we can use the addition formula and write multiplication in polar form:

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Definition. The addition formula also allows use to derive formulas for $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$. Thus

$$\cos n\theta + i\sin n\theta = e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$$

By equating $\cos n\theta$ with the real terms and $\sin n\theta$ with the imaginary terms, we produce identites that are known as **de Moivre's formulae**.

Definition. A complex number z is an $\boldsymbol{n}$**th root** of w if $z^n = w$. The nth roots of w are precisely the roots of the polynomial $z^n - w$. The roots are given explicitly by

$$\begin{aligned} r &= \rho^{1/n} \\ \theta &= \frac{\varphi}{n} + \frac{2\pi k}{n}, \qquad k = 0, 1, 2, \dots, n-1 \end{aligned}$$

Graphically, the roots are distributed equally around the circle centered at 0 with radius $|w|^{1/n}$.

Definition. The nth roots of 1 are also called the $\boldsymbol{n}$**th roots of unity** and are given explicitly by:

$$\omega_k = e^{2\pi ik/n}, \qquad 0 \le k \le n-1$$

1.3 Stereographic Projection

This section was skipped.

1.4 The Square and Square Root Functions

Definition. The square function, $w = z^2$, is better understood in polar form. From the polar decomposition $w = z^2 = r^2 e^{2i\theta}$, we have

$$|w| = |z|^2,$$
$$\arg w = 2 \arg z$$

Definition. Finding the inverse function for $w = z^2$ is more difficult. Every number $w \neq 0$ is mapped by exactly two values of z, the square roots $\pm\sqrt{w}$. In order to define an inverse function, we must restrict the domain in the z-plane so that each w is mapped to by exactly one z.

All the values in the right half the z-plane map to the entire w-plane. Thus we can draw a **slit** or **branch cut** along the negative real axis from $-\infty$ to 0, and we can define the inverse function on the **slit plane** $\mathbb{C} \setminus (-\infty, 0]$.

Definition. We refer to the determination of the inverse function as the **branch** of the inverse. One branch $f_1(w)$ of the inverse function is defined by declaring $f_1(w)$ the value z such that $\operatorname{Re} z > 0$ and $z^2 = w$. Then $f_1(w)$ maps the slit plane $\mathbb{C} \setminus (-\infty, 0]$ onto the right half of the z-plane. To specify $f_1(w)$ explicitly, express $w = \rho e^{i\varphi}$ where $-\pi < \varphi < \pi$, and then

$$f_1(w) = \sqrt{\rho} e^{i\varphi/2}$$

The function f_1 is called the **principal branch** of $\sqrt{w}$. It is expressed in terms of the argument function as

$$f_1(w) = |w|^{1/2} e^{i(\operatorname{Arg} w)/2}, \qquad w \in \mathbb{C} \setminus (-\infty, 0]$$

The branch $f_2(w)$ is defined similarly except it maps values in the w-plane to values on the left half of the z-plane. The two slit planes, corresponding to $f_1(w)$ and $f_2(w)$ form the **Riemann surface** of $\sqrt{w}$.

1.5 The Exponential Function

Definition. We extend the exponential function to all complex numbers z be defining

$$e^z = e^x \cos y + i e^x \sin y, \qquad z \in \mathbb{C}$$

Since $e^{iy} = \cos y + i \sin y$, we could write

$$e^z = e^x e^{iy}$$

This identity is simply the polar representaton of $w = e^z$:

$$|w| = |e^x|,$$
$$\arg w = y$$

Since $\cos x$ and $\sin y$ are 2π-periodic, e^z is also 2π-periodic:

$$e^{z+2\pi i} = e^z, \qquad z \in \mathbb{C}$$

Additional properties of the exponential function

$$e^{z+w} = e^z e^w, \qquad z, w \in \mathbb{C}$$
$$1/e^z = e^{-z}, \qquad z \in \mathbb{C}$$

1.6 The Logarithm Function

Definition. For $z \neq 0$, we define $\log z$ to be a multi-valued function

$$\begin{aligned}\log z &= \log|z| + i \arg z \\ &= \log|z| + i\text{Arg}\, z + 2\pi i m, \qquad m = 0, \pm 1, \pm 2, \ldots\end{aligned}$$

The values of $\log z$ are precisely the complex numbers w such that $e^w = z$.

Definition. We define the **principle value of** $\log z$ to be

$$\text{Log}\, z = \log|z| + i\text{Arg}\, z, \qquad z \neq 0$$

Thus $\text{Log}\, z$ is a single-valued inverse of e^w with values in the horizontal strip $-\pi < \text{Im}\, w \leq \pi$. From $\text{Log}\, z$ we can find the other values of $\log z$:

$$\log z = \text{Log}\, z + 2\pi i m, \qquad m = 0, \pm 1, \pm 2, \ldots$$

1.7 Power Functions and Phase Factors

Definition. Let α be an arbitrary complex number. We defined the power function z^α to be the multivalued function

$$z^\alpha = e^{\alpha \log z}, \qquad z \neq 0$$

Thus the values of z^a are

$$\begin{aligned}z^\alpha &= e^{\alpha[\log|z| + i\text{Arg}\, z + 2\pi i m]} \\ &= e^{\alpha \text{Log}\, z} e^{2\pi i \alpha m}, \qquad m = 0, \pm 1, \pm 2, \ldots\end{aligned}$$

Definition. If α is not an integer, we cannot define z^α on the complex plane such that the function is continuous. We must make a branch cut and consider the continuous branch of z^α defined explicitly on the slit plane $\mathbb{C} \setminus [0, \infty)$ by

$$w = r^\alpha e^{i\alpha\theta}, \qquad \text{for } z = re^{i\theta}, \quad 0 < \theta < 2\pi$$

For $\theta = 0$, we have $z^\alpha = r^\alpha$. For $\theta = 2\pi$, we have $z^\alpha = r^\alpha e^{2\pi i \alpha}$. We call the multiplier $e^{2\pi i \alpha}$ the **phase factor** of z^α at $z = 0$.

Phase Change Lemma. Let $g(z)$ be a single-valued function that is defined and continuous near z_0. For any continuously varying branch of $(z - z_0)^\alpha$, the function $f(z) = (z - z_0)^\alpha g(z)$ is multiplied by the phase factor $e^{2\pi i \alpha}$ when z traverses a complete circle about z_0 in the positive direction.

1.8 Trigonometric and Hyperbolic Functions

We extend the definition of $\sin z$ and $\cos z$ to complex numbers by using their exponential forms:

$$\begin{aligned}\sin z &= \frac{e^{iz} + e^{-iz}}{2}, \qquad z \in \mathbb{C} \\ \cos z &= \frac{e^{iz} - e^{-iz}}{2i}, \qquad z \in \mathbb{C}\end{aligned}$$

These definitions agree with the usual definition when z is real. Evidently, $\cos z$ is still an even function and $\sin z$ is still an odd function,

$$\cos(-z) = \cos z, \qquad z \in \mathbb{C}$$
$$\sin(-z) = -\sin z, \qquad z \in \mathbb{C}$$

They are still 2π-periodic,

$$\cos(z + 2\pi) = \cos z, \qquad z \in \mathbb{C}$$
$$\sin(z + 2\pi) = \sin z, \qquad z \in \mathbb{C}$$

The addition formulae remain valid,

$$\cos(z + w) = \cos z \cos w - \sin z \sin w, \qquad z \in \mathbb{C}$$
$$\sin(z + w) = \sin z \cos w - \cos z \sin w, \qquad z \in \mathbb{C}$$

And the following identity still holds true,

$$\cos^2 z + \sin^z = 1, \qquad z \in \mathbb{C}$$

Definition. We define the hyperbolic functions in a similar manner.

$$\sinh z = \frac{e^z + e^{-z}}{2}, \qquad z \in \mathbb{C}$$
$$\cosh z = \frac{e^z - e^{-z}}{2}, \qquad z \in \mathbb{C}$$

Both $\cosh z$ and $\sinh z$ are periodic with period $2\pi i$,

$$\cosh(z + 2\pi i) = \cosh z, \qquad z \in \mathbb{C}$$
$$\sinh(z + 2\pi i) = \sinh z, \qquad z \in \mathbb{C}$$

When viewed as functions of complex variables, the trigonometric and hyperbolic functions exhibit a close relationship. They are obtained from each other by rotating the domain space by $\pi/2$,

$$\cosh(iz) = \cos z, \qquad \cos(iz) = \cosh z$$
$$\sinh(iz) = i \sin z, \qquad \sin(iz) = i \sinh z$$

Using these equations and the addition formula for $\sin z$, we obtain the Cartesian representation for $\sin z$,

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \qquad z = x + iy \in \mathbb{C}$$

Thus,

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

And using $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y = 1 + \sinh y$, we obtain

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

The other trigonometric and hyperbolic functions are obtained from their usual formulae:

$$\tan z = \frac{\sin z}{\cos z} \qquad \tanh z = \frac{\sinh z}{\cosh z}, \qquad z \in \mathbb{C}$$

Definition. The inverse trigonometric functions are mutlivalued functions that can be expressed in terms of the logarithm function. Suppose $w = \sin^{-1} z$. Then solving

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i} = z$$

we obtain

$$\sin^{-1} z = -i \log\left(iz \pm \sqrt{1 - z^2}\right)$$

The other functions can be obtained in a similar manner.

2 Analytic Functions

2.1 Review of Basic Analysis

Definition. A sequence of complex numbers $\{s_n\}$ **converges** to s if for any $\epsilon > 0$, there is an integer $N \geq 1$ such that $|s_n - s| < \epsilon$ for all $n \geq N$. If $\{s_n\}$ convergs to s, we write $s_n \to s$ or $\lim s_n = s$.

Definition. A sequence of complex numbers $\{s_n\}$ is said to be **bounded** if there is some finite number $R > 0$ such that $|s_n| < R$ for all n.

Theorem. Suppose $\{s_n\}$ and $\{t_n\}$ are bounded sequences such that $s_n \to s$ and $t_n \to t$, then
(a) $s_n + t_n \to s + t$
(b) $s_n t_n \to st$
(c) $s_n/t_n \to s/t,$ provided that $t \neq 0$

Theorem. A sequence $\{s_n\}$ of complex numbers converges if and only if the corresponding sequences of real and imaginary parts of the s_n's converge.

Definition. We define a sequence of complex numbers $\{s_n\}$ to be a **Cauchy sequence** if the differences $s_n - s_m$ tend to 0 as n and m tend to ∞. More formally, a sequence is Cauchy if for any $\epsilon > 0$, there exists an $N \geq 1$ such that $|s_n - s_m| < \epsilon$ if $m, n \geq N$.

Theorem. A sequence is complex numbers converges if and only if it is a Cauchy sequence.

Definition. We say a that a complex-valued function $f(z)$ **has limit** L **as** z **tends to** z_0 if the valued $f(z)$ are near L whenever z is near z_0, $z \neq z_0$. More formally, $f(z)$ has limit L as z tends to z_0 if for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$. In this case, we write

$$\lim_{z \to z_0} f(z) = L,$$

or $f(z) \to L$ as $z \to z_0$.

Lemma. The complex-valued function $f(z)$ has limit L as $z \to z_0$ if and only if $f(z_n) \to L$ for any sequence $\{z_n\}$ in the domain of $f(z)$ such that $z_n \to z_0$ and $z_n \to z_0$.

Theorem. If a function has a limit at z_0, then the function is bounded near z_0. Futher, if $f(z) \to L$ and $g(z) \to M$ as $z \to z_0$, then the following are true as $z \to z_0$:
(a) $f(z) + g(z) \to L + M$
(b) $f(z)g(z) \to LM$
(c) $f(z)/g(z) \to L/M,$ provided that $M \neq 0$

Definition. We say that $f(z)$ is **continuous at** z_0 if $f(z) \to f(z_0)$ as $z \to z_0$. A **continuous function** is a function that is continuous at every point of its domain.

Definition. A subset D of the complex plane is a **domain** if D is open and any two points of D can be connected by a broken line segment within D.

Theorem. If $h(x, y)$ is continuous differentiable on a domain D such that $\nabla h = 0$ on D, then h is constant.

Definition. A set is **convex** if whenever two points belong to the set, the straight line segment joining them is contained within the set. A set is **star-shaped with respect to** z_0 if whenever a point belongs to the set, the straight line segment between it and z_0 is contained within the set. Any convex set if star-shaped with respect to each of its points. A **star-shaped domain** is a domain that is star-shaped with respect to one of its points.

Definition. A subset E of the complex plane is **closed** if it contains the limit of any convergent subsequence in E. The **boundary** of a set E consists of the points z such that every disk centered at z contains both points in E and not in E. A subset of the complex plane is said to be **compact** if it is both closed and bounded.

Theorem. A continuous real-valued function on a compact set attains a maximum and a minimum.

2.2 Analytic Functions

Definition. A complex-valued function $f(z)$ is **differentiable** at z_0 if the difference quotients

$$\frac{f(z) - f(z_0)}{z - z_0}$$

have a limit as $z \to z_0$. The limit is denoted by $f'(z_0)$ or by $\frac{df}{dz}(z_0)$, and we refer to it as the **complex derivative** of $f(z)$ at z_0. Thus

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

It is often useful to write the difference quotient in the form

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

so that $z - z_0$ is replaced by Δz. Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Theorem. If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

The complex derivative satisfies the usual rules of differentiating sums, products and quotients.

$$\begin{aligned}
(cf)'(z) &= cf'(z) \\
(f + g)'(z) &= f'(z) + g'(z) \\
(fg)'(z) &= f(z)g'(z) + f'(z)g(z) \\
(f/g)'(z) &= \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}
\end{aligned}$$

Theorem (Chain Rule). Suppose that $g(z)$ is differentiable at z_0, and suppose that $f(w)$ is differentiable at $w_0 = g(z_0)$. Then the composition $(f \circ g)(z) = f(g(z))$ is differentiable at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0)$$

Alternatively, we can write the chain rule as

$$\frac{df}{dz} = \frac{df}{dw}\frac{dw}{dz}$$

Definition. A function $f(z)$ is **analytic** on the open set U if $f(z)$ is differentiable at each point of U and the complex derivative $f'(z)$ is continuous on U.

2.3 The Cauchy-Riemann Equations

Theorem. Let $f = u + iv$ be defined on a domain D in the complex plane, where u and v are real-valued. Then $f(z)$ is analytic on D if and only if $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives that satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations are called the **Cauchy-Riemann equations** for u and v.

Theorem. If $f(z)$ is analytic on a domain D and $f'(z) = 0$ on D, then $f(z)$ is constant.

Theorem. If $f(z)$ is analytic and real-valued on a domain D, then $f(z)$ is constant.

2.4 Inverse Mappings and the Jacobian

Definition. Let $f = u + iv$ be analytic on a domain D. We may regard D as a domain in the Euclidean plane $\mathbb{R}^2$ and f as a mapping from D to $\mathbb{R}^2$ with components $(u(x, y), v(x, y))$. The **Jacobian matrix** of this map is

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

and its determinant is

$$\det J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Theorem. If $f(z)$ is analytic, then its Jacobian matrix (as a map of R^2) has determinant

$$\det J_f = |f'(z)|^2$$

Theorem. Suppose $f(z)$ is analytic on a domain D, $z_0 \in D$, and $f'(z_0) \neq 0$. Then there is an disk $U \subset D$ containing z_0 such that $f(z)$ is one-to-one on U, the image $V = f(U)$ of U is open, and the inverse function

$$f^{-1} : V \to U$$

is analytic and satisfies

$$(f^{-1})'(f(z)) = 1/f'(z), \qquad z \in U.$$

2.5 Harmonic Functions

Definition. The equation

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

is called **Laplace's equation**. The operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial}{\partial x_n^2}$$

is called the **Laplacian**. In terms of the operator, Laplace's equation is simply $\Delta u = 0$. Smooth functions $u(x_1, \ldots, x_n)$ that satisfy Laplace's equation are called **harmonic functions**. For complex functions, we will only be concerned about the solutions of the equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

Definition. We say that a function $u(x, y)$ is **harmonic** if all of its first- and second-order partial derivatives exist and are continuous and satisfy Laplace's equation.

Theorem. If $f = u + iv$ is analytic, and the functions u and v have continuous second-order partial derivatives, then u and v are harmonic.

Definition. If u is harmonic on a domain D, and v is a harmonic function such that $f = u + iv$ is analytic, we say that v is a **harmonic conjugate** of u. The harmonic conjugate is unique up to adding a constant.

Theorem. Let D be an open disk, or an open rectangle with sides parallel to the axes, and let $u(x, y)$ be an harmonic function on D. Then there is a harmonic function $v(x, y)$ on D such that $f = u + iv$ is analytic on D. The harmonic conjugate v is unique, up to a constant.

2.6 Conformal Mappings

Let $\gamma(t) = x(t) + iy(t)$, $0 \leq t \leq 1$, be a smooth parameterized curve terminating at $z_0 = \gamma(0)$. We refer to

$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}$$

as the **tangent curve** to the curve γ at z_0. It is the complex representation of the usual tangent vector. We define the **angle between two curves** at z_0 to be the angle between their tangent vectors at z_0.

Theorem. If $\gamma(t)$, $0 \leq t \leq 1$, is a smooth parameterized curve terminating at $z_0 = \gamma(0)$, and $f(z)$ is analytic, then the tangent to the curve $f(\gamma(t))$ terminating at $f(z_0)$ is

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$$

Definition. A complex-valued function $g(z)$ is **conformal** at z_0 if wherenever γ_0 and γ_1 are two curves terminating at z_0 with nonzero tangents, then the curves $g \circ \gamma_0$ and $g \circ \gamma_1$ have nonzero tangents at $g(z_0)$ and the angle from $(g \circ \gamma_0)'(z_0)$ to $(g \circ \gamma_1)'(z_0)$ is the same as the angle from $\gamma_0'(z_0)$ to $\gamma_1'(z_0)$. A **conformal mapping** of one domain D onto another V is a continuously differentiable function that is conformal at each point of D and that maps D one-to-one onto V.

Theorem. If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

2.7 Fractional Linear Transformations

Definition. A **fractional linear transformation** is a function of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex constants satisfying $ad - bc \neq 0$. The transformations are also called **Möbius transformations**.

Definition. A function of the form $f(z) = ax + b$ where $a \neq 0$ is called an **affine transformation**. Special cases of affine transformations are **translations** $f(z) = z + b$ and **dilations** $f(z) = az$. Meanwhile, the transformation $f(z) = 1/z$ is called an **inversion**. All of these examples are fractional linear transformations.

Theorem. Every fractional linear transformations is a composition of dilations, translations, and inversion.

Theorem. A fractional linear transformation maps circles in the extended complex plane to circles.

3 Line Integrals and Harmonic Function

3.1 Line Integrals and Green's Theorem

Definition. A **path** in the plane from A to B is a continuous function $t \mapsto \gamma(t)$ on some parameter interval $a \leq t \leq b$. such that $\gamma(a) = A$ and $\gamma(b) = B$. The path is **simple** if $\gamma(s) \neq \gamma(t)$ when $s \neq t$. The path is **closed** if it starts and ends at the same point, that is, $\gamma(a) = \gamma(b)$. A **simple closed path** is a closed path γ such that $\gamma(s) \neq \gamma(t)$ for $a \leq s < t < b$.

Definition. If $\gamma(t)$, $a \leq t \leq b$ is a path from A to B and if $\phi(s)$, $\alpha \leq s \leq \beta$, is a strictly increasing continuous function satisfying $\phi(\alpha) = a$ and $\phi(\beta) = b$, then the composition $\gamma(\phi(s))$, $\alpha \leq s \leq \beta$, is also a path from A to B. The composition $\gamma \circ \phi$ is a **reparameterization** of γ.

Definition. The **trace** of a path γ is its image $\gamma([a, b])$, which is a subset of the plane.

Definition. A **smooth path** is a path that can be represented in the form $\gamma(t) = (x(t), y(t))$, $a \leq t \leq b$ where the functions $x(t)$ and $y(t)$ are smooth. A **piecewise smooth path** is a concatenation of smooth paths. A **curve** is a (usually) smooth or piecewise smooth path.

Definition. Let γ be a smooth path on the complex plane and let $P(x, y)$ and $Q(x, y)$ be continuous complex-valued functions. We consider successive points along the path and form the sum

$$\sum P(x_j, y_j)(x_{j+1} - x_j) + \sum Q(x_j, y_j)(y_{j+1} - y_j)$$

If these sums have a limit as the distance between the succesive point tend to 0, we define the limit to be the **line integral** of $P\,dx + Q\,dy$ along γ and denote it by

$$\int_\gamma P\,dx + Q\,dy$$

Definition. A domain D has **piecewise smooth boundary** if the boundary of D can be decomposed into a finite number of smooth curves meeting only at the endpoints. We denote the boundary of D by ∂D. For the purposes of integration, the **orientation of** D is chosen so that D lies on the left of a curve in ∂D as we traverse the boundary curve in the positive direction.

Theorem (Green's Theorem). Let D be a bounded domain in the plane whose boundary ∂D consists of a finite number of disjoint piecewise smooth closed curves. Let P and Q be continuously differentiable functions on $D \cup \partial D$. Then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dx\,dy$$

3.2 Independence of Path

Definition. If $h(x, y)$ is a continuously differentiable complex-valued function, we define the **differential** dh of h by

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy$$

We say that a differential $P\,dx + Q\,dy$ is **exact** if $P\,dx + Q\,dy = dh$ for some function h.

Fundamental Theorem of Calculus, Part I. If γ is a piecewise smooth curve from A to B, and if $h(x, y)$ is continuously differentiable on γ, then

$$\int_\gamma dh = h(B) - h(A)$$

Definition. Let P and Q be continuous complex-valued functions on a domain D. We say that a line integral $\int P\,dx+Q\,dy$ is **independent of path** in D if for any two points A and B in D, the integrals $\int_\gamma P\,dx+Q\,dy$ are the same for any path γ between A and B. This is equivalent to saying $\int_\gamma P\,dx+Q\,dy = 0$ for any closed path in D.

Lemma. Let P and Q be continuous complex-valued functions on a domain D. Then $\int P\,dx + Q\,dy$ is independent of path in D if and only if $P\,dx + Q\,dy$ is exact, that is, there is a continuously differentiable function $h(x, y)$ such that $dh = P\,dx + Q\,dy$. Moreover, the function h is unique, up to adding a constant.

Definition. We say that a differential is **closed** on D if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thus Green's Theorem implies that if $P\,dx+Q\,dy$ is closed on D, then $\int_{\partial U} P\,dx+Q\,dy = 0$ for any bounded domain D with piecewise smooth boundary such that U is contained in D.

Lemma. Exact differentials are closed.

Fundamental Theorem of Calculus, Part II. Let P and Q be continuously differentiable complex-valued functions on a domain D. Suppose

(i) D is a star-shaped domain, and
(ii) the differential $P\,dx + Q\,dy$ is closed on D.

Then $P\,dx + Q\,dy$ is exact on D.

In general,

$$\text{independent of path} \iff \text{exact} \implies \text{closed}$$

while for star-shaped-domains,

$$\text{independent of path} \iff \text{exact} \iff \text{closed}$$

Theorem. Let D be a domain, and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \le t \le b$, be two paths in D from A to B. Suppose γ_0 can be continually deformed to γ_1, in the sense that for $0 \le s \le 1$ there are paths $\gamma_s(t)$, $a \le t \le b$, from A to B such that $\gamma_s(t)$ depends continuously on s and t for $0 \le s \le 1$, $a \le t \le b$. Then

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$$

for any closed differential $P\,dx + Q\,dy$ on D.

Theorem. Let D be a domain, and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \le t \le b$, be two closed paths in D. Suppose γ_0 can be continually deformed to γ_1, in the sense that for $0 \le s \le 1$ there are paths $\gamma_s(t)$, $a \le t \le b$, such that $\gamma_s(t)$ depends continuously on s and t for $0 \le s \le 1$, $a \le t \le b$. Then

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$$

for any closed differential $P\,dx + Q\,dy$ on D.

3.3 Harmonic Conjugates

Lemma. If $u(x, y)$ is harmonic, then the differential

$$-\frac{\partial u}{\partial y}\,dy + \frac{\partial v}{\partial x}\,dy$$

is closed.

Theorem. Any harmonic function $u(x, y)$ on a star-shaped domain D (as a disk or rectangle) has a harmonic conjugate function $v(x, y)$ on D.

3.4 The Mean Value Property

Definition. Let $h(z)$ be a continuous real-valued function on a domain D. Let $z_0 \in D$, and suppose D contains the disk $\{|z - z_0| < \rho\}$. We define the **average value** of $h(z)$ on the circle $\{|z - z_0| = r\}$ to be

$$A(r) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})\,d\theta, \qquad 0 < r < \rho$$

Theorem. If $u(z)$ is a harmonic function on a domain D, and if the disk $\{|z - z_0| < \rho\}$ is contained in D, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})\,d\theta, \qquad 0 < r < \rho$$

Definition. We say that a continuous function $h(z)$ on a domain D has the **mean value property** if for every point $z_0 \in D$, $h(z_0)$ is the average of its value over any small circle centered at z_0. More formally, for any $z_0 \in D$, there is an $\epsilon > 0$ such that

$$h(z_0) = \int_0^{2\pi} h(z_0 + re^{i\theta})\,d\theta, \qquad 0 < r < \epsilon$$

Harmonic functions satisfy the mean value property. The converse is also true: continuous functions that satisfy the mean value property are harmonic functions (Chapter X).

3.5 The Maximum Principle

Strict Maximum Principle (Real Version). Let $u(z)$ be a real-valued harmonic function on a domain D such that $u(z) \le M$ for all $z \in D$. If $u(z_0) = M$ for some $z_0 \in D$, then $u(z) = M$ for all $z \in D$.

Strict Maximum Principle (Complex Version). Let $h(z)$ be a complex-valued harmonic function on a domain D such that $|h(z)| \leq M$ for all $z \in D$. If $|h(z_0)| = M$ for some $z_0 \in D$, then $h(z)$ is constant on D.

Maximum Principle. Let $h(z)$ be a complex-valued harmonic function on a bounded domain D such that $h(z)$ extends continuously to the boundary ∂D of D. If $|h(z)| \leq M$ for all $z \in \partial D$, then $|h(z)| \leq M$ for all $z \in D$.

3.6 Applications to Fluid Dynamics

This section was skipped.

3.7 Other Applications to Physics

This section was skipped.

4 Complex Integration and Analyticity

4.1 Complex Line Integrals

For complex analysis, it is convenient to define $dz = dx + i\,dy$. According, to this notation, if $h(z)$ is a complex-valued function on a curve γ, then

$$\int_\gamma h(z)\,dz = \int_\gamma h(z)\,dx + i\int_\gamma h(z)\,dy$$

Additionally, we define the infinitesimal arc length ds by $|dz|$:

$$|dz| = ds = \sqrt{(dx)^2 + (dy)^2}$$

This means that if a curve γ is parameterized by $z(t) = x(t) + iy(t)$, then

$$\int_\gamma h(z)\,|dz| = \int_\gamma h(z)\,ds = \int_a^b h(z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dy}\right)^2}$$

In particular, the length of γ is

$$L = \int_\gamma |dz| = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dy}\right)^2}$$

Theorem. Suppose γ is a piecewise smooth curve. If $h(z)$ is a continuous function on γ, then

$$\left|\int_\gamma h(z)\,dz\right| \leq \int_\gamma |h(z)|\,|dz|$$

Further if γ has length L and $h(z) \leq M$ on γ, then

$$\left|\int_\gamma h(z)\,dz\right| \leq ML$$

The equation above is called the **ML-estimate**.

4.2 Fundamental Theorem of Calculus for Analytic Functions

Definition. Let $f(z)$ be a continuous function on a domain D. A function $F(z)$ on D is a **(complex) primitive** for $f(z)$ if $F(z)$ is analytic and $F'(z) = f(z)$.

Theorem (Part I.). If $f(z)$ is continuous on a domain D, and if $F(z)$ is a primitive for $f(z)$, then

$$\int_A^B f(z)\, dz = F(B) - F(A)$$

where the integral can be taken over any path in D from A to B.

Theorem (Part II.). Let D be a star-shaped domain and let $f(z)$ be analytic on D. Then $f(z)$ has a primitive on D, and the primitive is unique up to adding a constant. A primitive for f is given explicitly by

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta, \qquad z \in D$$

where z_0 is any fixed point of D, and where the integral can be taken along any path in D from z_0 to z.

4.3 Cauchy's Theorem

Theorem. A continuously differentiable function $f(z)$ on D is analytic if and only if the differential $f(z)\, dz$ is closed.

Theorem (Cauchy's Theorem). Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to ∂D, then

$$\int_{\partial D} f(z)\, dz = 0$$

4.4 Cauchy Integral Formula

Theorem (Cauchy Integral Formula). Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and $f(z)$ extends smoothly to the boundary of D, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z}\, dw \qquad z \in D$$

Theorem. Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and $f(z)$ extends smoothly to the boundary of D, then $f(z)$ has complex derivatives of all orders on D, which are given by

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z)^{m+1}}\, dw \qquad z \in D, m \geq 0$$

Corollary. If $f(z)$ is analytic on a domain D, then $f(z)$ is infinitely differentiable, and the successive complex derivatives $f'(z), f''(z), \ldots$ are all analytic on D.

4.5 Liouville's Theorem

Theorem (Cauchy's Estimates). Suppose $f(z)$ is analytic for $|z - z_0| \leq \rho$. If $f(z) \leq M$ for $|z - z_0| = \rho$, then

$$\left| f^{(m)}(z_0) \right| \leq \frac{m!}{\rho^m} M, \qquad m \geq 0$$

Definition. We define an **entire function** to be a function that is analytic on the entire complex plane.

Theorem (Liouville's Theorem). A bounded entire function is constant.

4.6 Morera's Theorem

Theorem (Morera's Theorem). Let $f(z)$ be a continuous function on a domain D. If $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangle R in D with sides parallel to the coordinate axis, then $f(z)$ is analytic on D.

Theorem. Suppose that $h(t, z)$ is a continuous, complex-valued function, defined for $a \leq t \leq b$ and $z \in D$. If for each fixed t, $h(t, z)$ is an analytic function of $z \in D$, then

$$H(z) = \int_a^b h(t, z)\,dz, \qquad z \in D,$$

is analytic on D.

Theorem. Suppose that $h(z)$ is a continuous function on a domain D that is analytic on $D \setminus \mathbb{R}$, that is, on the part of D not lying on the real axis. Then $f(z)$ is analytic on D.

4.7 Goursat's Theorem

Theorem (Goursat's Theorem). If $f(z)$ is a complex-valued function on a domain D such that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for each point z_0 of D, then $f(z)$ is analytic on D.

4.8 Complex Notation and Pompeiu's Formula

Definition. Many results in complex analysis can be expressed in terms of the first-order differential equations.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

Thus, we can think of $\partial f/\partial z$ as an average of the derivatives of $f(z)$ in the x and in the iy directions. When deriving the Cauchy-Riemann equations, we derived two equations for $f'(z)$

$$f'(z) = \frac{\partial f}{\partial x} \qquad f'(z) = -i\frac{\partial f}{\partial y}$$

Taking the average of these expressions, we get

$$f'(z) = \frac{\partial f}{\partial z}$$

assuming that $f(z)$ is analytic. If we let $f = u + iv$, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

Therefore, we obtain the equation

$$\frac{\partial f}{\partial \overline{z}} = 0$$

which is equivalent to the Cauchy-Riemann equations. This equation is called the **complex form of the Cauchy-Riemann equations**.

Theorem. Let $f(z)$ be a continuously differentiable function on a domain D. Then $f(z)$ is analytic if and only if $f(z)$ satisfies that complex form of the Cauchy-Riemann equations. If $f(z)$ is analytic, then the derivative of $f(z)$ is given by $\partial f/\partial z$.

Theorem. Let $f(z)$ be a continuously differentiable function on a domain D. Suppose that the gradient of $f(z)$ does not vanish at any point on D, and that $f(z)$ is conformal. Then $f(z)$ is analytic on D, and $f'(z) \neq 0$ on D.

Theorem. If D is a bounded domain in the complex plane with piecewise smooth boundary, and if $g(z)$ is a smooth function on $D \cup \partial D$, then

$$\int_{\partial D} g(z)\,dz = 2i \iint_D \frac{\partial g}{\partial \overline{z}}\,dx\,dy$$

Theorem (Pompeiu's Formula). Suppose D is a bounded domain in the complex plane with piecewise smooth boundary. If $g(z)$ is a smooth complex-valued function on $D \cup \partial D$, then

$$g(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{g(z)}{z - w}\,dz - \frac{1}{\pi}\iint_D \frac{\partial g}{\partial \overline{z}}\frac{1}{z - w}\,dx\,dy, \qquad w \in D$$

5 Power Series

5.1 Infinite Series

Definition. A series $\sum_{k=0}^{\infty} a_k$ of complex numbers is said to **converge** to S if sequence of partial sums $S_k = a_0 + \ldots + a_k$ converges to S. We denote the sum S by $\sum_{k=0}^{\infty} a_k$ or $\sum a_k$. Any statement about series is just a statement about sequences. Thus if $\sum a_k = A$ and $\sum b_k = B$, then $\sum(a_k + b_k) = A + B$ and $\sum ca_k = cA$.

Theorem (Comparison Test). If $0 \leq a_k \leq r_k$, and if $\sum r_k$ converges, then $\sum a_k$ converges and $\sum a_k \leq \sum r_k$.

Theorem. If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

Definition. The series $\sum a_k$ is said to **converge absolutely** if $\sum |a_k|$ converges.

Theorem. If $\sum a_k$ converges absolutely, then $\sum a_k$ converges, and

$$\left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k|$$

5.2 Sequences and Series of Functions

Definition. Let $\{f_j\}$ be a sequence of complex-valued functions defined on some set E. We set that the sequence $\{f_j\}$ **converges pointwise** on E if for each point $x \in E$, the sequence of complex numbers $\{f_j(x)\}$ converges. The limit $f(x)$ of $\{f_j(x)\}$ is then a complex-valued function on E.

Definition. We set that the sequence $\{f_j\}$ of functions on E **converges uniformly** to f on E if $|f_j(x) - f(x)| \leq \epsilon_j$ for all $x \in E$, where $\epsilon_j \to 0$ as $j \to \infty$. We can think of ϵ_j as the worst-case estimator of the difference $|f_j(x) - f(x)|$, that is $\epsilon_j = \sup |f_j(x) - f(x)|$. Note that if $\{f_j\}$ converges uniformly to f on E, then $\{f_j\}$ converges pointwise to f on E.

Theorem. Let $\{f_j\}$ be a sequence of complex-valued functions defined on a subset E of the complex plane. If each f_j is continuous on E and $\{f_j\}$ converges uniformly to f on E, then f is continuous on E.

Theorem. Let γ be a piecewise smooth curve in the complex plane. If $\{f_j\}$ is a sequence of complex-valued functions on γ, and $\{f_j\}$ converges uniformly to f on γ, then $\int_\gamma f_j(z)\,dz$ converges to $\int_\gamma f(z)\,dz$.

Theorem (Weierstrass M-Test). Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(z)$ are complex-valued functions on a set E such that $|g_k(x)| \leq M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E.

Theorem. If $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges uniformly to $f(z)$ on D, then $f(z)$ is analytic on D.

Theorem. Suppose that $f_k(z)$ is analytic for $|z - z_0| \leq R$, and suppose that the sequence $\{f_k(z)\}$ converges uniformly to $f(z)$ for $|z - z_0| \leq R$. Then for each $r < R$ and for each $m \geq 1$, the sequence of mth derivatives $\{f_k^{(m)}(z)\}$ converges uniformly to $f^{(m)}(z)$ for $|z - z_0| \leq r$.

Definition. We say that a sequence $\{f_k(z)\}$ of analytic functions on a domain D **converges normally** to the analytic function $f(z)$ on D if it converges uniformly to $f(z)$ on each closed disk contained in D.

Theorem. Suppose that $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to the analytic functions $f(z)$. Then for each $m \geq 1$, the sequence of mth derivatives $\{f_k^{(m)}(z)\}$ converges normally to $f^{(m)}(z)$ on D.

5.3 Power Series

Definition. A **power series** (centered at z_0) is a series of the form $\sum_{k=0}^{\infty} a_k(z - z_0)^k$.

Theorem. Let $\sum a_k z^k$ be a power series. Then there is R, $0 \leq R \leq \pm\infty$, such that $\sum a_k z^k$ converges absolutely if $|z| < R$, and $\sum a_k z^k$ does not converge if $|z| < R$. For each fixed r satisfying $r < R$, the series $\sum a_k z^k$ converges uniformly for $|z| \leq r$.

Definition. We call R the **radius of convergence** of the series $\sum a_k z^k$.

Theorem. Suppose $\sum a_k z^k$ is a power series with radius of convergence $R > 0$. Then the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad |z| < R$$

is analytic. The derivatives of $f(z)$ are obtained by differentiating the series term by term.

Theorem (Ratio Test). If $|a_k/a_{k+1}|$ has a limit as $k \to \infty$, either finite or $+\infty$, then the limit is the radius of convergence R of $\sum a_k z^k$,

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

Theorem (Root Test). If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, either finite or $+\infty$, then the radius of convergence R of $\sum a_k z^k$ is given by

$$R = \frac{1}{\lim \sqrt[k]{|a_k|}}.$$

Definition. There is a more general formula for the Root Test called the **Cauchy-Hadamard formula** that gives the radius of convergence for any power series in terms of the lim sup,

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}$$

5.4 Power Series Expansion of an Analytic Function

Theorem. Suppose that $f(z)$ is analytic for $|z - z_0| < \rho$. Then $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad |z - z_0| < \rho$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!} \qquad k \geq 0$$

and where the power series has radius of convergence $R \geq \rho$. For any fixed r, $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta, \qquad k \geq 0$$

Further if, $|f(z)| \leq M$ for $|z - z_0| = r$, then

$$|a_k| \leq \frac{M}{r^k}, \qquad k \geq 0$$

Corollary. Suppose that $f(z)$ and $g(z)$ are analytic on $|z - z_0| < r$. If $f^{(k)}(z_0) = g^{(k)}(z_0)$ for $k \geq 0$, then $f(z) = g(z)$ for $|z - z_0| < r$.

Corollary. Suppose that $f(z)$ is analytic at z_0 with power series expansion $f(z) = \sum a_k (z - z_0)^k$ centered at z_0. Then the radius of convergence of the power series is the largest number R such that $f(z)$ can be extends to be analytic on the disk $\{|z - z_0| < R\}$.

5.5 Power Series Expansions at Infinity

This section was skipped.

5.6 Manipulation of Power Series

This section was skipped.

5.7 The Zeros of an Analytic Function

Definition. Let $f(z)$ be analytic at z_0 and suppose $f(z_0) = 0$. We say that $f(z)$ has a **zero of order** N at z_0 if

$$f(z_0) = f'(z_0) = \ldots + f^{(N-1)}(z_0) = 0$$

If we write $f(z)$ using its power series representation,

$$f(z_0) = a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \ldots$$

then we can factor out $(z - z_0)^N$ and write

$$f(z_0) = (z - z_0)^N h(z)$$

where $h(z)$ is analytic at z_0 and $h(z_0) = a_N \neq 0$. Conversely, if there is a factorization where $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$, then the leading term in the power series for $f(z)$ is $h(z_0)(z-z_0)^N$, and $f(z)$ has a zero of order N at z_0. A zero of order one is called a **simple zero**, and a zero of order two is called a **double zero**.

Definition. We say that a point $z_0 \in E$ is an **isolated point** of the set E if there is $\rho > 0$ such that $|z - z_0| \geq \rho$ for all points in E other than z_0. In other words, z_0 is an isolated point of E if z_0 is a positive distance from $E \setminus \{z_0\}$. If E is a set such that every point of E is an isolated point of E, we say that the points of E are isolated.

Theorem. If D is a domain, and $f(z)$ is an analytic function on D that is not identically zero, then the zeros of $f(z)$ are isolated.

Theorem (Uniqueness Principle). If $f(z)$ and $g(z)$ are analytic on a domain D, and if $f(z) = g(z)$ for all z belonging to a set that has nonisolated point, then $f(z) = g(z)$ for all $z \in D$.

Theorem. Let D be a domain, and let E be a subset of D that has a nonisolated point. Let $F(z, w)$ be a function defined for $z, w \in D$ such that $F(z, w)$ is analytic in z for each fixed $w \in D$ and analytic in w for each fixed $z \in D$. If $F(z, w) = 0$ whenever z and w both belong to E, then $F(z, w) = 0$ for all $z, w \in D$.

5.8 Analytic Continuation

This section was skipped.

6 Laurent Series and Isolated Singularities

6.1 The Laurent Decomposition

Theorem (Laurent Decomposition). Suppose $0 \leq \rho < \sigma \leq +\infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. Then $f(z)$ can be decomposed as a sum

$$f(z) = f_0(z) + f_1(z)$$

where $f_0(z)$ is analytic for $|z - z_0| < \sigma$, and $f_1(z)$ is analytic for $|z - z_0| > \rho$ and at ∞.

Definition. Choose r and s such that $\rho < r < s < \sigma$. By Cauchy's Integral Formula, the formulas of the functions in Laurent Decomposition are given by

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

which is valid for $r < |z - z_0| < s$, and

$$f_0(z) = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad |z - z_0| < s,$$

$$f_1(z) = -\frac{1}{2\pi i}\oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad |z - z_0| > r,$$

Definition. If a function $f(z)$ can be decomposed using Laurent Decomposition. Then we can expression the function using its **Laurent series expansion**

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \qquad \rho < |z - z_0| < \sigma$$

which converges uniformly and absolutely on $r \leq |z - z_0| \leq s$ where $\rho < r < s < \sigma$.

Theorem. Suppose $0 \leq \rho < \sigma \leq \infty$ and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. Then $f(z)$ has a Laurent series expansion that converges absolutely at each point on the annulus and converges uniformly on each closed subannulus $r \leq |z - z_0| < s$ where $\rho < r < s < \sigma$. The coefficients are uniquely determined by $f(z)$ for any fixed r, $\rho < r < \sigma$ and are given by

$$a_n = \frac{1}{2\pi i}\oint_{|z - z_0| = r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \qquad -\infty < n < \infty$$

6.2 Isolated Singularities of an Analytic Function

Definition. A point z_0 is an **isolated singularity** of $f(z)$ if $f(z)$ is analytic in some punctured disk $\{0 < |z - z_0| < r\}$ cenetered at z_0. Suppose that $f(z)$ has an isolated singularity at z_0. Then $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \qquad 0 < |z - z_0| < r$$

Definition. The isolated singularity of $f(z)$ at z_0 is defined to be **removable singularity** if $a_k = 0$ for all $k < 0$. In this case the Laurent series becomes a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad 0 < |z - z_0| < r$$

If we define $f(z_0) = a_0$, the function $f(z)$ becomes analytic on the entire disk $\{|z - z_0| < r\}$.

Theorem (Riemann's Theorem on Removable Singularities). Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_0, then $f(z)$ has a removable singularity at z_0.

Definition. The isolated singularity of $f(z)$ at z_0 is defined to be a **pole** if there is $N > 0$ such that $a_{-N} \neq 0$, but $a_k = 0$ for all $k < -N$. The integer N is the **order** of the pole. In the case the Laurent series becomes

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k, \qquad 0 < |z - z_0| < r$$

A pole of order one is called a **simple pole**, and a pole of order two is called a **double pole**.

Theorem. Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $f(z) = g(z)/(z - z_0)^N$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Theorem. Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $1/f(z)$ is analytic at z_0 and has a zero of order N.

Definition. We say that a function $f(z)$ is **meromorphic** on a domain D if $f(z)$ is analytic on D except possibly at isolated singularities, each of which is a pole. A meromorphic function f at z_0 is said to have **order** N at z_0 if $f(z) = (z - z_0)^N g(z)$ for some analytic function g at z_0 such that $g(z_0) \neq 0$. The order of the function 0 is defined to be $+\infty$.

Theorem. Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Definition. The isolated singularity of $f(z)$ at z_0 is defined to be an **essential singularity** if $a_k \neq 0$ for infinitely many $k < 0$. Thus an isolated singularity that is neither removable nor a pole is declared to be essential.

Theorem (Casorati-Weierstrass Theorem). Suppose z_0 is an essential isolated singularity of $f(z)$. Then for every complex number w_0, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$.

6.3 Isolated Singularity at Infinity

This section was skipped.

6.4 Partial Fraction Decomposition

This section was skipped.

6.5 Periodic Fractions Decomposition

This section was skipped.

6.6 Fourier Series

This section was skipped.

7 The Residue Calculus

7.1 The Residue Theorem

Definition. Suppose z_0 is an isolated singularity of $f(z)$ and that $f(z)$ has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad 0 < |z - z_0| < \rho$$

We define the **residue** of $f(z)$ at z_0 to be the coefficient a_{-1} of $1/(z - z_0)$ in this Laurent expansion,

$$\text{Res}\,[f(z), z_0] = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\,dz$$

where r is any fixed radius satisfying $0 < r < \rho$.

Theorem (Residue Theorem). Let D be a bounded domain in the complex plane with piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$, except for a finite number of isolated singularities $z_1, \ldots, z_m$ in D. Then

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \text{Res}\,[f(z), z_j]$$

Rule 1. If $f(z)$ has a simple pole at z_0, then

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

Rule 2. If $f(z)$ has a double pole at z_0, then

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} \frac{d}{dz} [(z - z_0)^2 f(z)]$$

Rule 3. If $f(z)$ and $g(z)$ are analytic at z_0, and if $g(z)$ has a simple zero at z_0, then

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

Rule 4. If $g(z)$ is analytic and has a simple zero at z_0, then

$$\text{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$$

7.2 Integrals Featuring Rational Functions

The Residue Theorem can be used to evaluate integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dz$$

For the integral to converge, $P(z)$ and $Q(z)$ must be polynomials and $Q(z)$ has no zeroes on the real axis. It is also required that

$$\deg Q(z) \geq \deg P(z) + 2$$

Then evaluating the integral on the half-disk in the upper half-plane and letting the radius go to ∞, we have

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dz = 2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$$

Integrals of rational functions with a trigonometric multiplier can also be computed using Residue Theorem. For example,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\, dx = \pi e^{-a}$$

Here, we replace $\cos(ax)$ with $e^{iz} = e^{-y}$ which is bounded above in magnitude by 1 in the upper half-plane.

7.3 Integral of Trigonometric Functions

Integrals with polar coordinates can be converted into a line integral on a disk in the complex plane. We use the following parameterization

$$d\theta = \frac{dz}{iz}$$

for the differential and the exponential forms of $\sin z$ and $\cos z$.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}$$

Therefore,

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \oint_{|z|=1} \frac{1}{a + \frac{1}{2}(z + 1/z)} \frac{dz}{iz} = \frac{2}{i} \oint_{|z|=1} \frac{dz}{z^2 + 2az + 1}$$

can be computed using the residue theorem.

7.4 Integrands with Branch Points

This section was skipped.

7.5 Fractional Residues

This section was skipped.

7.6 Principal Values

This section was skipped.

7.7 Jordan's Lemma

This section was skipped.

7.8 Exterior Domains

This section was skipped.

8 The Logarithmic Integral

8.1 The Argument Principle

Definition. Suppose $f(z)$ is analytic on a domain D. For a curve γ in D such that $f(z) \neq 0$ on γ, we refer to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma d\log f(z)$$

as the **logarithmic integral** of $f(z)$ along γ. Thus the logarithmic integral measures the change of $\log f(z)$ along the curve γ. It can be used to count zeros and poles of meromorphic functions.

Theorem. Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities.

Definition. Evaluating the logarithmic integral yields

$$\frac{1}{2\pi i}\int_\gamma d\log f(z) = \frac{1}{2\pi i}\int_\gamma d\log|f(z)| + \frac{1}{2\pi}\int_\gamma d\arg(f(z))$$

The differential $d\log|f(z)|$ is exact. If we parameterize γ by $\gamma(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_\gamma d\log|f(z)| = \log|f(\gamma(b))| - \log|f(\gamma(a))|$$

depends solely on $\gamma(a)$ and $\gamma(b)$. In particular, the integral is 0 on any closed curve. The differential $d\arg f(z)$ is closed but not exact. Integrating on γ gives us

$$\int_\gamma d\arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$$

This quantity is referred to as the **increase in the argument of** $f(z)$ **along** γ. It is defined for any path γ in D providing there are no zeros on poles on the path. If a bounded domain D has a boundary ∂D consists of a finite number of piecewise-smooth curves, then the **increase in the argument of** $f(z)$ **around the boundary of** D to be the sum of its increase around the closed curves in ∂D.

Theorem. Let D be a bounded domain with piecewise smooth boundary ∂D, and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then the increase in the argument on $f(z)$ around the boundary of D is 2π times the number of zeros minus the number of poles of $f(z)$ in D,

$$\int_{\partial D} d\arg(f(z)) = 2\pi(N_0 - N_\infty)$$

8.2 Rouché's Theorem

Theorem (Roché's Theorem). Let D be a bounded domain with piecewise smooth boundary ∂D. Let $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D, counting multiplicities.

8.3 Hurwitz's Theorem

Theorem (Hurwitz's Theorem). Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$, and suppose that $f(z)$ has a zero of order N at z_0. Then there exists $\rho > 0$ such that for large k, $f_k(z)$ has exactly N zeros in the disk $\{|z - z_0| < \rho\}$ counting multiplicity, and these zeros converge to z_0 as $k \to \infty$.

Definition. We say that a function is **univalent** on a domain D if it is analytic and one-to-one on D.

Theorem. Suppose $\{f_k(z)\}$ is a sequence of univalent functions on a domain D that converges normally on D to a function $f(z)$. Then either $f(z)$ is univalent or $f(z)$ is constant.

8.4 Open Mapping and Inverse Function Theorems

Theorem (Open Mapping Theorem for Analytic Functions.). If $f(z)$ is analytic on a domain D, and $f(z)$ is not constant, then $f(z)$ maps open sets to open sets, that is $f(U)$ is open for each open subset of D.

Theorem (Inverse Function Theorem). Supoose $f(z)$ is analytic for $|z - z_0| \leq \rho$ and satisfies $f(z_0) = w_0$, $f'(z_0) \neq 0$, and $f(z) \neq w_0$ for $0 < |z - z_0| \leq \rho$. Let $\delta > 0$ be chosen such that $|f(z) - w_0| \geq \delta$ for $|z - z_0| = \rho$. Then for each w such that $|w - w_0| < \delta$, there is a unique z satisfying $|z - z_0| < \rho$ and $f(z) = w$. Writing $z = f^{-1}(w)$, we have

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w} \, d\zeta, \qquad |w - w_0| < \delta$$

8.5 Critical Points

This section was skipped.

8.6 Winding Numbers

Definition. Let $\gamma(t), a \leq t \leq b$ be a closed path in D. We define the **trace of** γ to be the image $\Gamma = \gamma([a, b])$ of γ. For $z_0 \neq \Gamma$, we define the **winding number** $W(\gamma, z_0)$ of γ around z_0 to be the increase in the argument of $z - z_0$ around γ, normalized by dividing by 2π. If γ is piecewise smooth, the winding number is the integer

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi} \int_\gamma d \arg(z - z_0), \qquad z_0 \neq \Gamma$$